Rewriting meets Homotopy

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Categorification

Weak structures are ubiquitous

- Monoidal categories, Equivalence of categories
- Bicategories, ∞-groupoids, ∞-categories
- A_∞- , E_∞-algebras
- ∞-topos, cartesian closed bicategories

Categorification

monoid	monoidal category, A_∞-algebra
isomorphism	equivalence of category
groupoids	∞-groupoids
categories	bicategories, ∞-categories
commutative monoids	symmetric/braided monoidal categories, E_∞- algebras
topos	∞-topos
cartesian closed categories	cartesian closed bicategories

Goals

- There is a general theory of "weakening" of structures encoded by operads.
- **Ex:** From $\mathrm{Set} \subset \mathrm{Cat}$, we get
monoids $\rightsquigarrow$ monoidal category.
- **Warning / Advertisement:** Many structures cannot be encoded by operads! (e.g. semilatices, groups, ...)

Main questions

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  Any (weak) monoidal category is equivalent to a strict one: what is all the fuss about?
- What is a *good* weakening of a structure?
  Why do we impose the pentagon and triangle axioms?
- How to compute a weakened version of my favorite structure?
  How could we come up with the pentagon and triangle axioms?

From monoids to monoidal categories

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The first arrow stems from the inclusion of sets into categories (or modules into chain complexes).

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The first arrow stems from the inclusion of sets into categories (or modules into chain complexes).

- monoid actions (on sets) $\rightsquigarrow$ monoid actions on groupoids $\rightsquigarrow$ weak monoid actions

Monoid action

Take a monoid M acting on a groupoid $\mathcal{C}$.

- For any $m \in M$, $\bar{m} : \mathcal{C} \to \mathcal{C}$,

$$\bar{m} \circ \bar{n} = \overline{mn} \qquad \bar{1} = id$$

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There should be an action of M on $\mathcal{D}$!

$$\begin{tikzcd} \bar{m} \arrow[r, phantom] & \mathcal{C} \arrow[loop left] \arrow[rr, bend left, "F"] \arrow[rr, phantom, "\simeq"] & & \mathcal{D} \arrow[ll, bend left, "G"] \end{tikzcd}$$

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- Define $\bar{m}(x) := F(\bar{m}(G(x)))$: this is *not* a monoid action!

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On $\mathcal{D}$, we only get natural isomorphisms:

$$\alpha^{m,n} : \bar{m} \circ \bar{n} \simeq \overline{mn} \qquad \beta : \bar{1} \simeq id$$

Those satisfy the following equation (+ equations involving β):

$$\begin{tikzcd} & \overline{m_1 m_2} \circ \bar{m}_3 \arrow[dr, "\alpha^{m_1 m_2, m_3}"] & \\ \bar{m}_1 \circ \bar{m}_2 \circ \bar{m}_3 \arrow[ur, "\alpha^{m_1,m_2} \circ \bar{m}_3"] \arrow[dr, "\bar{m}_1 \circ \alpha^{m_2,m_3}"'] & & \overline{m_1 m_2 m_3} \\ & \bar{m}_1 \circ \overline{m_2 m_3} \arrow[ur, "\alpha^{m_1, m_2 m_3}"'] & \end{tikzcd}$$

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Weak structures appear naturaly when studying objects up to equivalence

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Examples:

- Monoid actions on groupoids up to equivalence of categories
  $\rightsquigarrow$ weak monoid actions
- Monoid structures on categories (= strict monoidal categories) up to equivalence of categories
  $\rightsquigarrow$ monoidal categories
- Monoid structures on chain complexes (= dg-algebras) up to quasi-iso
  $\rightsquigarrow$ A_∞-algebras.

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What do we gain by considering weak structures?

- More examples (simply because any strict structure is also a weak one).
- *Better properties* w.r.t. the given notion of equivalence.

weak structures vs. higher structures

monoids $\rightsquigarrow$ strict monoidal categories $\rightsquigarrow$ monoidal categories

$$\mathrm{Set} \hookrightarrow \mathrm{Gpd}$$

$$\begin{tikzcd}
\mathrm{Op} = \mathrm{Op}(\mathrm{Set}) \arrow[r, hook] & \mathrm{Op}(\mathrm{Gpd}) \\
\mathrm{Mon} = \mathrm{Mon}(\mathrm{Set}) \arrow[r, hook] \arrow[u, hook] & \mathrm{Mon}(\mathrm{Gpd}) \arrow[u, hook]
\end{tikzcd}$$

Action of monoids $\subseteq$ Action of strict monoidal categories.

A reformulation of weak actions

Action of M on $\mathcal{C}$ $\Longleftrightarrow$ Monoidal functor $M \to [\mathcal{C}, \mathcal{C}]$
Define a strict monoidal groupoid $\tilde{M}$ as follows:

- Objects: the free monoid on elements of M,
- Arrows: freely generated by arrows $\alpha^{m,n} : m \otimes n \to mn$ and $\beta : 1_M \to I_{\tilde{M}}$, up to the relations $m_1 \otimes m_2 \otimes m_3 \to m_1 m_2 \otimes m_3 \to m_1 m_2 m_3 = m_1 \otimes m_2 \otimes m_3 \to m_1 \otimes m_2 m_3 \to m_1 m_2 m_3$.

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Weak action of M on $\mathcal{C} \iff$ Action of $\tilde{M}$ on $\mathcal{C}$.

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Weak action of M on $\mathcal{C} \iff$ Action of $\tilde{M}$ on $\mathcal{C}$. Relationship between $\tilde{M}$ and M?

- There is a morphism $\pi : \tilde{M} \to M$ of strict monoidal groupoids

$$m_1 \otimes m_2 \otimes \ldots \otimes m_k \mapsto m_1 m_2 \ldots m_k \qquad \alpha^{m,n} \mapsto id_{mn} \qquad \beta \mapsto id_1$$

- which is an equivalence of groupoids.

Interpretation

- There is a morphism $\pi : \tilde{M} \to M$ of strict monoidal groupoids
- which is an equivalence of groupoids.

Faithfulness of π :

$$f, g : x \to y \in \tilde{M} \Longrightarrow f = g$$

This is a coherence theorem for monoid actions!

Proposition

There is an operad (in $\mathrm{Op}(\mathrm{Cat})$) $\widetilde{\mathrm{Mon}}$ such that monoidal categories are (strict!) algebras for $\widetilde{\mathrm{Mon}}$.

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Theorem (MacLane's coherence theorem)

In the category $\mathrm{Op}(\mathrm{Cat})$, the morphism $\widetilde{\mathrm{Mon}} \to \mathrm{Mon}$ is an equivalence of categories.

Freeness and stability under equivalence

- M algebras are not stable under equivalence... What about $\tilde{M}$-algebras?

Proposition

Let $\mathcal{M}$ be a strict monoidal groupoid. If the monoid of obects of $\mathcal{M}$ is free, then $\mathcal{M}$-actions are stable under equivalence.

Proof in the case $\mathcal{M} = E^*$:

- Take an action of E^* on $\mathcal{C}$ equivalent to $\mathcal{D}$.
- For any $e \in E$, this induces an endofunctor $\bar{e} : \mathcal{D} \to \mathcal{D}$.
- Extend this to an action of the free monoid.

Follow-up questions

- There can be many suitable $\tilde{M}$. What relationship between them?
- What is the relationship between M and $\tilde{M}$ algebras?
- In which context does all this make sense?

General framework

Given a monoidal category $\mathcal{C}$ (so far, $\mathcal{C} = (\mathrm{Gpd}, \times)$), we want:

- A notion of equivalence in $\mathcal{C}$.
- Define an equivalence of $\mathcal{C}$-operads as a morphism which induces an equivalence in $\mathcal{C}$.
- Suppose we have a notion of "good object" in $\mathcal{C}$-operads, whose algebras are stable under equivalence
- A weak algebra for a $\mathcal{C}$-operad $\mathcal{P}$ is an algebra over a "good operad" $\tilde{\mathcal{P}}$, equivalent to $\mathcal{P}$.

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Solution: $\mathcal{C}$ is equipped with a model category structure.

- Equivalences = weak equivalences / trivial fibrations.
- Under suitable hypothesis, it induces a model structure on $\mathrm{Op}(\mathcal{C})$.
- "Good objects" in $\mathrm{Op}(\mathcal{C})$ are the (Σ-)"cofibrant" objects.

General Theorems

Theorem (Berger-Moerdijk)

In good cases, the categories of algebra of two (Σ-)cofibrant replacements of an operad $\mathcal{P}$ have (Quillen-)equivalent categories of algebras.

Taking $\mathcal{P} = \mathrm{Ass}$ (resp. Com), a Σ-cofibrant replacement is called an A_∞-operad (resp. an E_∞-operad).

Corollary

Any monoidal category is equivalent to a strict monoidal category

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Corollary

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Theorem (Berger-Moerdijk)

Let $\mathcal{P}$ is (Σ-)cofibrant, and $f : X \to Y$ equivalence in $\mathcal{C}$. Under suitable hypothesis (on $\mathcal{C}$, f, X and/or Y), if X or Y is equipped with a $\mathcal{P}$-algebra structure, then we can transport this structure along f.

Taking a step back

What we have seen so far

- Weak structures are “better” replacement of strict ones.
- They are encoded by cofibrant replacements of operads.

How to compute this cofibrant replacement?

Back to monoids in Gpd

M a monoid, seen as a strict monoidal groupoid. A cofibrant replacement of M is the data of:

- a strict monoidal groupoid M_∞ and a morphism $\pi : M_\infty \to M$,
- such that the monoid of objects of M_∞ is free,
- the map π is surjective on objects,
- and it induces an equivalence of groupoids.

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A presentation of such an object is a triple $\langle E | R | \equiv \rangle$ where:

- A set E of generating objects,
- A set R of generating arrows $f : u \to v$, $u, v \in E^*$
- A congruence $\equiv$ between arrows generated from R.

$$M_\infty \qquad = \qquad E^* \Leftarrow (R^\leftrightarrow / \equiv)$$

Cofibrant replacements in $\mathrm{Mon}(\mathrm{Gpd})$

$$M_\infty \quad = \quad E^* \leftleftarrows (R^\leftrightarrow / \equiv)$$

To be a cofibrant replacement of M we need:

- A map $E \to M$ whose image is a generating subset of M,
- which induces an isomorphism $E^*/R^\leftrightarrow \to M$,
- such that for any $f, g : u \to v \in R^\leftrightarrow$, $f \equiv g$.

Example

- $E = M$
- $R = \{\alpha^{m,n} : m \otimes n \to mn, \beta : 1_M \to I\}$
- $$\alpha^{m_1 m_2, m_3} \circ (\alpha^{m_1, m_2} \otimes m_3) \equiv \alpha^{m_1, m_2 m_3} \circ (m_1 \otimes \alpha^{m_2, m_3})$$

Rk : (E, R) is none other than a presentation of M!

Can we start from an other presentation?

Given (E, R), we need to find a congruence $\equiv$ such that for any $f, g : u \to v \in R^{\leftrightarrow}$, $f \equiv g$.

- This is implied by an $\equiv$-compatible Church Rosser property!
- Equivalent to $\equiv$-compatible confluence,
- which can be reduced to termination and $\equiv$-compatible confluence of the critical pairs.

Theorem (Squier '94)

Let (E, R) be an oriented presentation of a monoid M. Suppose it is terminating and confluent. For any critical pair $(f : u \to v_1, g : u \to v_2)$, choose $f' : v_1 \to w, g' : v_2 \to w \in R^{\to}$.
Then the congruence induced by $f' \circ f \equiv g' \circ g$ defines a presentation $\langle E|R| \equiv\rangle$ of a cofibrant replacement of M.

Rk: Finite presentation $\implies$ Finite number of critical pairs

Some Examples

$$M = E^* = \langle e_1, \ldots, e_n | \emptyset \rangle$$

$$M_\infty = \langle e_1, \ldots, e_n | \emptyset | \emptyset \rangle$$

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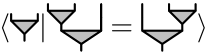
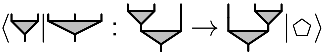
$P =$ 

$P_\infty =$ 

Further results

Computing cofibrant replacements by rewriting

- Case of Operads (/Pro/ProPs) due to Guiraud-Malbos.
- Monoids [Guiraud-Malbos, L.]: can be extended all the way to (strict) ω-groupoids ($\otimes$: Gray, folk model structure). Cofibrant $=$ Free. Generators in dim n: critical n-branchings

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- Case $\mathcal{C} = \mathrm{dgVect}$: weak equivalence = quasi-iso, cofibrant = projective.
  Algebras: origin in Gröbner basis. Anick resolution
  Operads: Dotsenko-Khoroshkin