Operads and Polytopes

I. Operads

1) What is an operad?
2) The operad As
3) The operad A_∞

II. Polytopes

4) What is a polytope?
5) The problem of the diagonal
6) A "mise en abyme" (my thesis)

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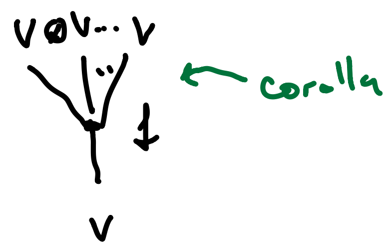
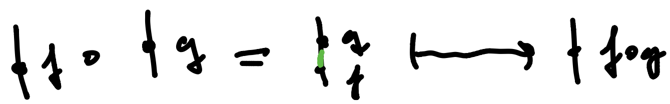
I. 1) $\mathbb{K}$ a field, $V \in \mathrm{Vect}_{\mathbb{K}}$. For each $n \in \mathbb{N}$

$$\mathrm{Hom}(V^{\otimes n}, V) =: \mathrm{End}_V(n)$$

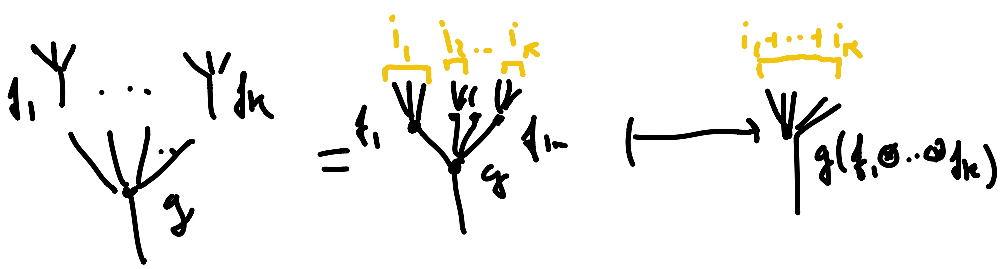
we represent f



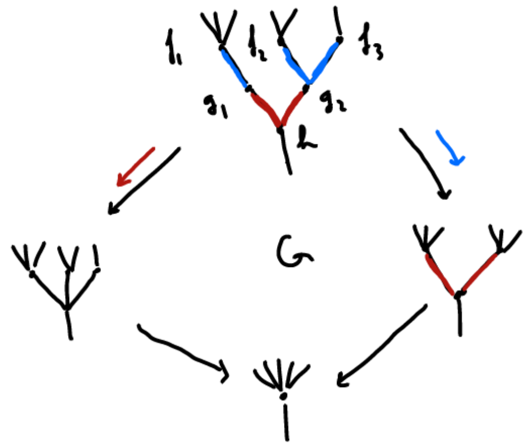
usual composition



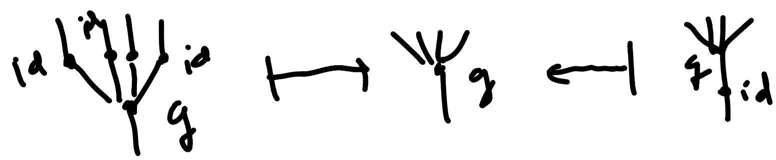
extends



This comp. is associative



We also have $\mathrm{id} \in \mathrm{End}_V(1)$



Def: A ns operad P is a family of v.s. $\{P(n)\}_{n\in\mathbb{N}}$ endowed with ass. and unit. <u>composite maps</u>

$$\gamma_{i_1,\dots,i_k} : P(k) \otimes P(i_1) \otimes \cdots \otimes P(i_k) \to P(i_1+\cdots+i_k)$$

Ex: $(A, \mu, 1_A)$ ass. alg

$P(1) := A \quad P(n) := 0 \ \forall n \neq 1, \ I := 1_A$ defines an operad

$$\longrightarrow \quad \{\text{ass. alg}\} \cong \left\{\begin{matrix}\text{operads in} \\ \text{arity } 1\end{matrix}\right\}$$

So 

ass. alg. $\longrightarrow$ operads

Def: A morphism of operads

$$\{ f_n : P(n) \longrightarrow Q(n) \}$$

pres. units & commuting with comp. maps.

Def: A P-alg is a morphism of operads

$$P \longrightarrow End_V \quad \text{for some } V \in Vect$$

In other words $$P(n) \otimes V^{\otimes n} \longrightarrow V$$

Q: What is the free P-alg?

$$V \longrightarrow \bigoplus_{n \geq 0} V^{\otimes n}$$

free ass. alg

$$V \xrightarrow{\text{Schur functor}} \bigoplus_{n \geq 0} P(n) \otimes V^{\otimes n}$$

Makes P into a monad in Vect.

Ex: $\begin{tikzcd} \mathrm{Ass} \arrow[d] & \mathrm{Com} \arrow[d] & \mathrm{Lie} \arrow[d] \\ \text{ass. alg} & \text{com. alg} & \text{Lie-alg} \end{tikzcd}$ $\Big\}$ "3 Graces" (Loday)

$V \otimes V \cdots V \circlearrowleft S_n$

f

V

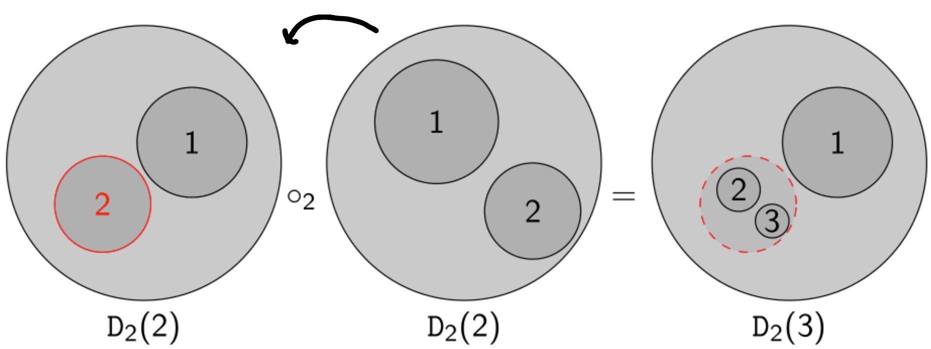
These are symmetric operads.

(Set, $\times$)

$\begin{tikzcd} \mathrm{Mon} := \{ \text{corolla with leaves } 1, 2, \dots, n,\ n \in \mathbb{N} \} \arrow[d] \\ \text{monoids} \end{tikzcd}$

(Top, $\times$) (dgVect, $\otimes$)

$\begin{tikzcd} D^2 \arrow[r, "H_\bullet"] \arrow[d] & \mathrm{Gerst} \arrow[d] \\ \Omega^2 X & \text{Gerst.-alg.} \end{tikzcd}$



2) The operad Ass {ass. alg}

$\not\simeq$

$\left\{ \text{operads in family} \right\}$ $\left\{ \text{alg. over an operad} \right\}$

Def: $As =$

0	0
1	$\mathbb{K}\,\vert$
2	$\mathbb{K}\,Y$
3	$\mathbb{K}\,\Psi$
4	$\mathbb{K}$
$\vdots$	$\vdots$

grafting

Composite maps

$$\gamma_{i_1,\dots,i_k} : As(k) \otimes As(i_1) \otimes \cdots \otimes As(i_k) \to As(i_1 + \cdots + i_k)$$

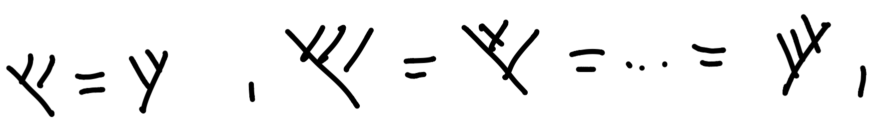
$$\mathbb{K} \otimes \mathbb{K} \otimes \cdots \otimes \mathbb{K} \xrightarrow{\cong} \mathbb{K}$$

scalar mult.

Def: $As := \mathcal{T}(Y) \big/ (\ \ - \ \ )$

free operad

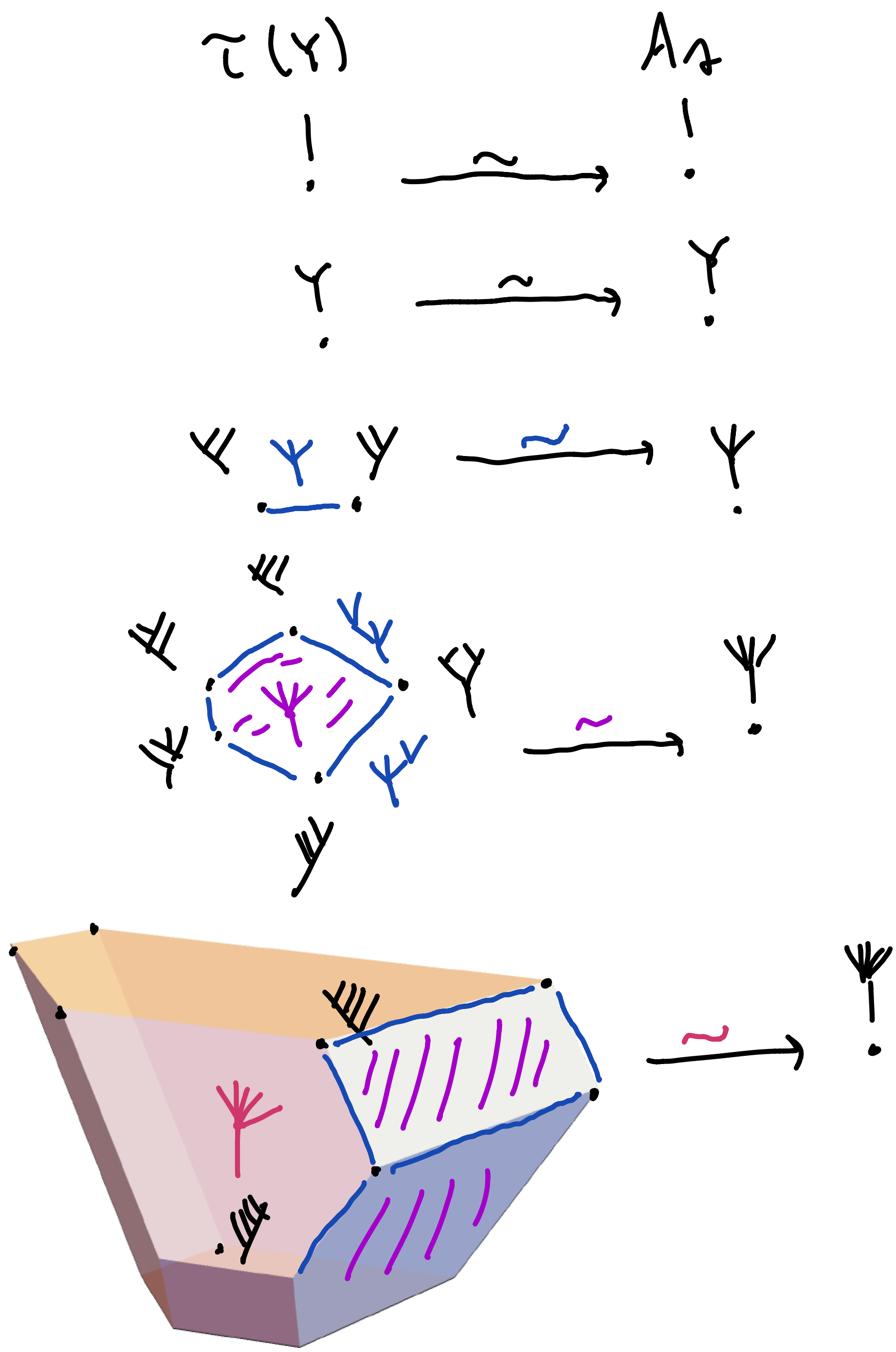
operadic ideal


, etc.

exercise: As-alg are ass. alg.

3) **Problem**: As-alg are <u>not</u> stable under htpy eq.

$\Rightarrow$ we need a replacement.



At each step we kill $(n-2)$th htpy gr.

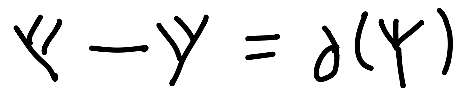
$\longrightarrow$ Associahedra $\{K_{n-2}\}_{n\geq 1}$

$$\mathcal{L}(K_{n-2}) \cong PT_n$$

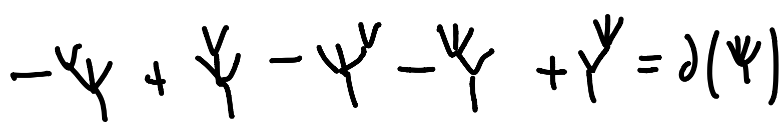
$\longrightarrow$ Quasi-free operad A_∞

$$A_\infty(n) := C_\bullet^{cell}(K_{n-2}) \cong \mathbb{k}[PT_n]$$

htpy associative



htpy between htpies

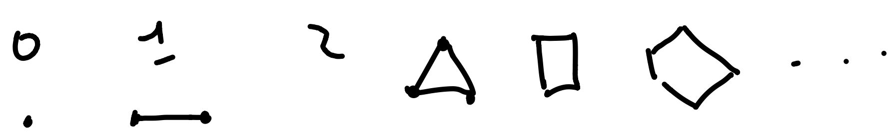


etc.

<u>THM</u>: A_∞-alg are stable under htpy eq. [Kad 80]

II. Polytopes

4) <u>Def</u>: A polytope is the convex hull of a finite set of pts in $\mathbb{R}^n$.



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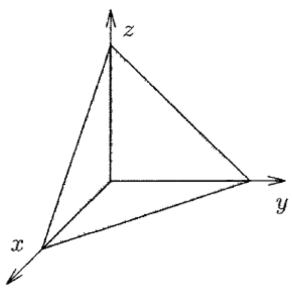
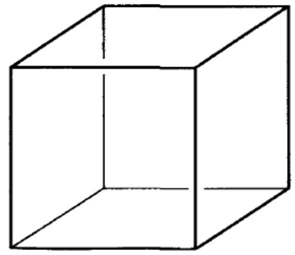
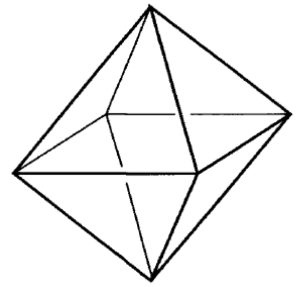
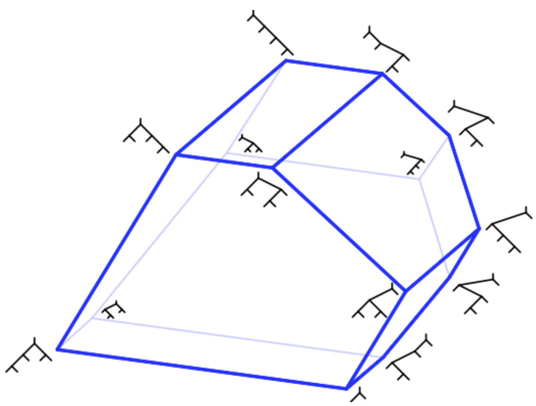
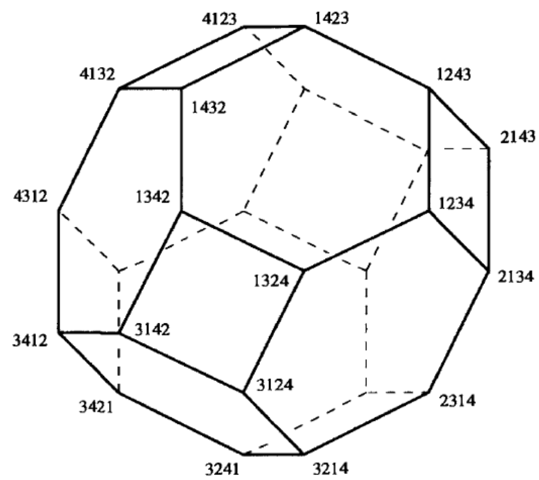




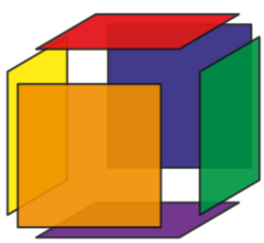


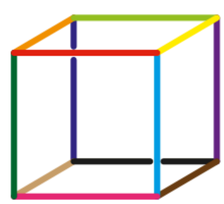


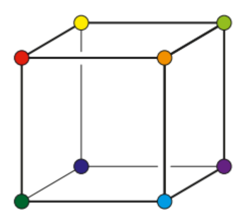


4 . . .

Def: The *faces* of a polytope P, denoted $\mathcal{L}(P)$,







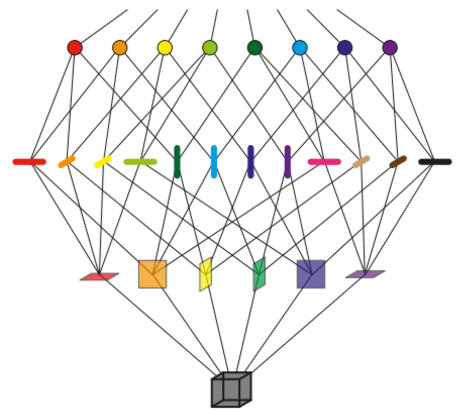
$\emptyset$

3 2 1 0 −1

THM: $(\mathcal{L}(P), \subset)$ is a *lattice*.

Def: $P \overset{\text{comb.}}{\sim} Q$ if

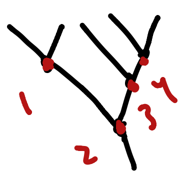
$$\mathcal{L}(P) \cong \mathcal{L}(Q)$$



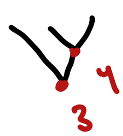
Def: A realization of the associahedron

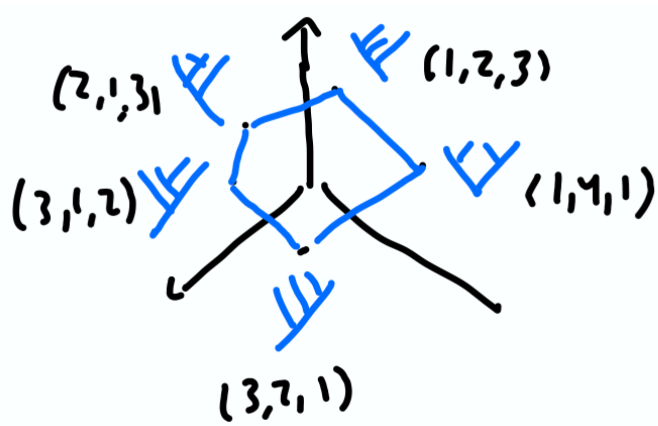
$$\mathcal{L}(P) \cong \{PT_n, \text{edge contr.}\}$$

Def: Loday realizations [Loday 04']

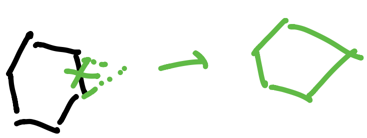


$\longrightarrow (1,6,2,1)$ in $\mathbb{R}^4$

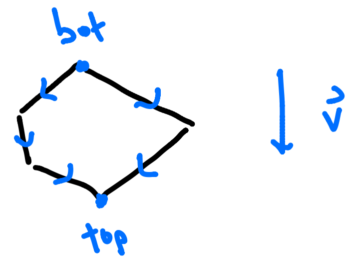


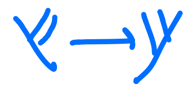


- int. coord. !
- removahedron



Def: $\vec{v} \in \mathbb{R}^n$ orients P if $\langle \vec{v}, \vec{e} \rangle \neq 0$ $\forall$ edges



Corresponds to successive appl. of the rewriting rule 

5) The diagonal

Problem: A, B A_∞-alg $\longrightarrow$ $A \otimes B$ A_∞-alg ?

We have

$$A_\infty \overset{?}{\dashrightarrow} A_\infty \underset{H}{\otimes} A_\infty \longrightarrow \mathrm{End}_A \underset{H}{\otimes} \mathrm{End}_B \cong \mathrm{End}_{A \otimes B}$$

Can be solved

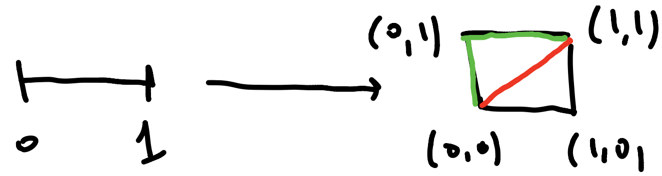
- "by hand" [Saneblidze-Umble 2004]
- conceptually at the polytope level

Problem: $\Delta_n : K_n \longrightarrow K_n \times K_n$

$$\begin{tikzcd} K_n \arrow[r] \arrow[d, "\text{cell}"] & K_n \times K_n \arrow[d] \\ A_\infty \arrow[r] & A_\infty \underset{H}{\otimes} A_\infty \end{tikzcd}$$

⚠ Δ_n has to be cellular!

⚠ <u>problem</u>:



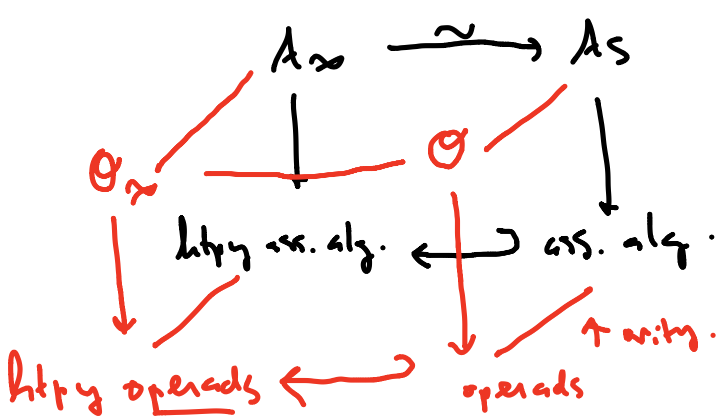
This problem was solved in 2019! [Masuda, Tonks, Thomas, Vallette]

* general method
* Loday realizations
* geometrically involved (orientation, "fractal" operad struct.)

<u>THM</u>: Magical formula $\mathrm{Im}\,\Delta_n = \bigcup_{\substack{F,G \\ \dim F + \dim G = n-2 \\ \mathrm{top}\, F \leq \mathrm{bot}\, G}} F \times G$

6) A "mise en abyme"

∃ colored coperad encoding ... ns operads!
(multicategory)



∃ a repl. $\mathcal{O}_\infty$ encoding ns operads up to htpy.

∃ associated family of polytopes.

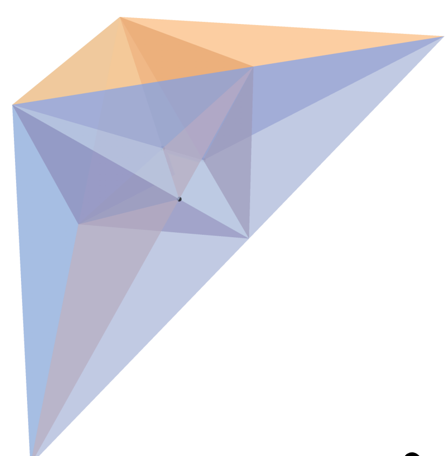
Goal of the project

1) associahedra ⟶ Loday real.

operahedra - - - → ?

ass. hemi-ass. perm.

2) colored op. struct. + diagonal ? yes.



3) Magical formula ? for ⊗ of operads up to htpy

work in progress...

Thank you for your attention!

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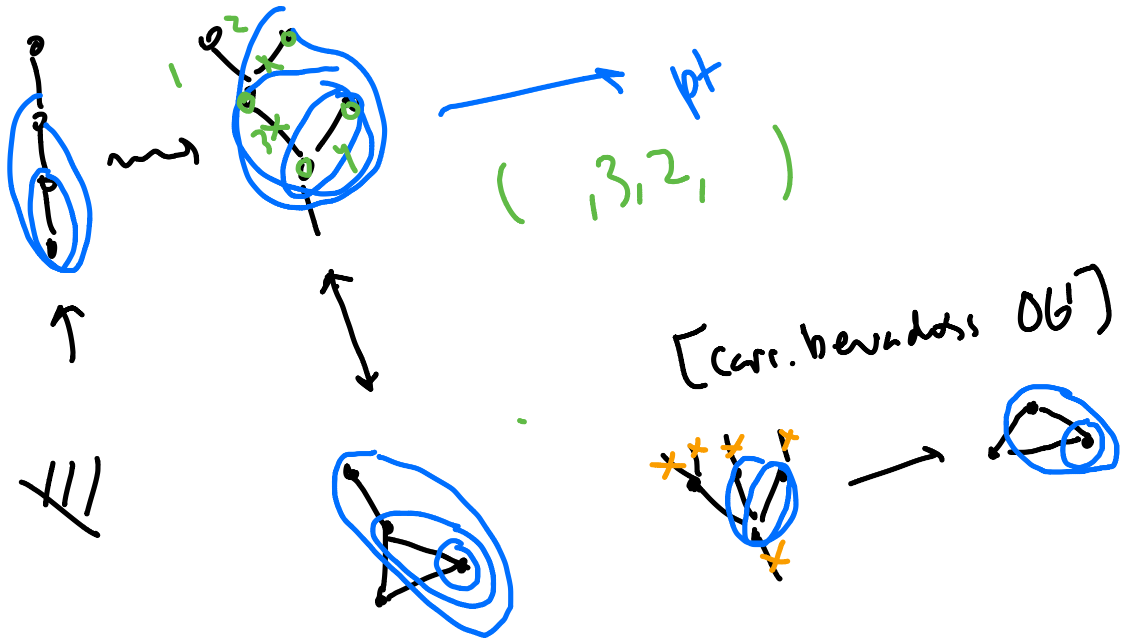

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